

Digital inversive vectors can achieve strong polynomial tractability for the weighted star discrepancy and for multivariate integration

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In memory of Joseph Frederick Traub (1932–2015)
and
Oscar Moreno de Ayala (1946–2015)

Abstract

We study high-dimensional numerical integration in the worst-case setting. The subject of tractability is concerned with the dependence of the worst-case integration error on the dimension. Roughly speaking, an integration problem is tractable if the worst-case error does not explode exponentially with the dimension. Many classical problems are known to be intractable. However, sometimes tractability can be shown. Often such proofs are based on randomly selected integration nodes. Of course, in applications true random numbers are not available and hence one mimics them with pseudorandom number generators. This motivates us to propose the use of pseudorandom vectors as underlying integration nodes in order to achieve tractability. In particular, we consider digital inverse vectors and present two examples of problems, the weighted star discrepancy and integration of Hölder continuous, absolute convergent Fourier- and cosine series, where the proposed method is successful.

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1 Introduction

We study numerical integration of multivariate functions f defined on the s -dimensional unit cube by means of quasi-Monte Carlo (QMC) rules, that is,

$$I_s(f) := \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x} \approx \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) =: Q_N(f),$$

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where we assume that $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ are well chosen integration nodes from the unit cube and where Q_N is the QMC-rule based on these nodes. General references for QMC integration are [5, 6, 15, 18].

Usually one studies integrands from a given Banach space $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ of functions. As quality criterion we consider the *worst-case error*

$$e(Q_N, \mathcal{F}) = \sup_{\substack{f \in \mathcal{F} \\ \|f\|_{\mathcal{F}} \leq 1}} |I_s(f) - Q_N(f)|.$$

For many function classes the problem of QMC integration is well-studied and one can achieve optimal asymptotic convergence rates for the worst-case error which are often of the form $O(N^{-1+\varepsilon})$ for $\varepsilon > 0$, with some implicit dependence on the dimension s . Although this is the best possible convergence rate in N , the dependence on the dimension can be crucial if s is large. This question is the subject of tractability. Tractability means that we control the dependence of the worst-case error on the dimension.

Tractability. For the numerical integration problem in \mathcal{F} the N th *minimal worst-case error* is defined as

$$e(N, s) = \inf_{A_{N,s}} \sup_{\substack{f \in \mathcal{F} \\ \|f\|_{\mathcal{F}} \leq 1}} |I_s(f) - A_{N,s}(f)|,$$

where the infimum is extended over all integration rules (not necessarily QMC rules) which are based on N function evaluations $f(\mathbf{x}_n)$, $n = 0, 1, \dots, N-1$. For $\varepsilon \in (0, 1)$ the *information complexity* $N(\varepsilon, s)$ is then defined as the minimal number of function evaluations which are required in order to achieve a worst-case error of at most ε . In other words¹,

$$N(\varepsilon, s) = \min\{N \in \mathbb{N} : e(N, s) \leq \varepsilon\}.$$

We say that we have:

- The *curse of dimensionality* if there exist positive numbers C , τ and ε_0 such that

$$N(\varepsilon, s) \geq C(1 + \tau)^s \quad \text{for all } \varepsilon \leq \varepsilon_0 \text{ and infinitely many } s \in \mathbb{N}.$$

- *Polynomial tractability* if there exist non-negative numbers C, τ_1, τ_2 such that

$$N(\varepsilon, s) \leq C s^{\tau_1} \varepsilon^{-\tau_2} \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1).$$

- *Strong polynomial tractability* if there exist non-negative numbers C and τ such that

$$N(\varepsilon, s) \leq C \varepsilon^{-\tau} \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1).$$

The *exponent τ^* of strong polynomial tractability* is defined as the infimum of τ for which strong polynomial tractability holds.

¹Here we speak about the absolute error criterion. Sometimes one uses the so-called initial error $e(0, s) = \|I_s\|$ as a reference value and asks for the minimal N in order to reduce this minimal error by a factor of ε . In this case one speaks about the normalized error criterion.

Besides polynomial and strong polynomial tractability there are many other notions of tractability such as *weak tractability*, which means that

$$\lim_{\varepsilon^{-1}+s \rightarrow \infty} \frac{\log N(\varepsilon, s)}{\varepsilon^{-1} + s} = 0.$$

Problems that are not weakly tractable (that is, the information complexity depends exponentially on ε^{-1} or s) are said to be *intractable*. The current state of the art in tractability theory is very well summarized in the three volumes of Novak and Woźniakowski [20, 21, 22].

It is known that many multivariate problems defined over standard spaces of functions suffer from the curse of dimensionality, as for example, integration of Lipschitz functions, monotone functions, convex functions (see [11]), or smooth functions (see [9, 10]). The reason for this disadvantageous behavior may be found in the fact that for standard spaces all variables and groups of variables are equally important. As a way out, Sloan and Woźniakowski [24] suggested to consider weighted spaces, in which the importance of successive variables and groups of variables is monitored by corresponding weights, to vanquish the curse of dimensionality and to obtain polynomial or even strong polynomial tractability, depending on the decay of the weights.

It is possible to construct QMC rules for some cases which achieve the one or other notion of tractability, for example (polynomial) lattice rules for integration in weighted Korobov spaces or Sobolev spaces. In such cases the point sets heavily depend on the chosen weights and can generally not be used for other weights. A further disadvantage is, that there is in general no explicit construction for point sets which can achieve tractability error bounds and thus one relies on computer search algorithms (for example, the fast component-by-component constructions, see [5, 15] and the references therein).

On the other hand, for those instances of problems which are tractable this property is often proved with randomly selected point sets. A particular example is the “inverse of star discrepancy problem” for which Heinrich, Novak, Wasilkowski, and Woźniakowski [12] showed with the help of random point sets that the star discrepancy is polynomially tractable (see also [1]). In [2] it is even shown that with very high probability (say 99%), a randomly selected point set satisfies the aforementioned bounds. However, if we generate a random point set on a computer using a pseudorandom number generator, this result does not apply since the pseudo-random numbers are deterministically constructed. Thus a fundamental question is whether known pseudo-random generators can be used to generate point sets which satisfy discrepancy bounds which imply polynomial tractability.

Thus in this paper we consider point sets generated by a certain pseudorandom number generator as candidates for point sets which achieve tractability for certain problems. First attempts in this direction have been made in [4, 8] where so-called p -sets are used (see also the survey [7]).

In the present paper we consider another choice of pseudorandom numbers obtained from explicit inversive pseudorandom number generators. We show that such point sets can be used to achieve tractability for two problems, namely the weighted star discrepancy problem (Section 3) and integration of functions from a subclass of the Wiener algebra which has some additional smoothness properties (Section 4).

In the subsequent section we introduce the proposed pseudorandom vectors and prove an estimate on an exponential sum from which we can derive discrepancy- and worst-case error bounds.

2 Explicit inversive vectors

Let \mathbb{F}_q be the finite field of order $q = p^k$ with a prime p and an integer $k \geq 1$. Further let $\{\beta_1, \dots, \beta_k\}$ be an ordered basis of \mathbb{F}_q over \mathbb{F}_p .

From a finite vector set in \mathbb{F}_q^s

$$\{\mathbf{z}_n = (z_{n,1}, z_{n,2}, \dots, z_{n,s}) \in \mathbb{F}_q^s : n = 0, 1, \dots, N-1\},$$

we can derive a point set in the s -dimensional unit interval. More precisely, if

$$\mathbf{z}_n = \mathbf{c}_n^{(1)}\beta_1 + \mathbf{c}_n^{(2)}\beta_2 + \dots + \mathbf{c}_n^{(k)}\beta_k \quad (1)$$

with all $\mathbf{c}_n^{(j)} = (c_{n,1}^{(j)}, c_{n,2}^{(j)}, \dots, c_{n,s}^{(j)}) \in \mathbb{F}_p^s$, then we define an s -dimensional *digital point set*

$$\mathcal{P}_s = \left\{ \mathbf{x}_n = \sum_{j=1}^k \mathbf{c}_n^{(j)} p^{-j} \in [0, 1)^s : n = 0, 1, \dots, N-1 \right\}. \quad (2)$$

The following point set was essentially introduced in [19].

Definition 1 (Set of explicit inversive points of size q). Put

$$\bar{z} = \begin{cases} z^{-1} & \text{if } z \in \mathbb{F}_q^*, \\ 0 & \text{if } z = 0. \end{cases}$$

For $1 \leq s \leq q$ we choose a subset $S \subseteq \mathbb{F}_q$ of cardinality s . We consider the vector set

$$\mathcal{S} = \{\mathbf{z}_0, \dots, \mathbf{z}_{q-1}\} = \{(\overline{u+v})_{v \in S} : u \in \mathbb{F}_q\} \subset \mathbb{F}_q^s \quad (3)$$

of size q and derive $\mathcal{P}_s = \{\mathbf{x}_0, \dots, \mathbf{x}_{q-1}\} \in [0, 1)^s$ from \mathcal{S} by (1) and (2). Note that here $N = |\mathcal{P}_s| = q$.

Our second point set was essentially introduced in [23] and is defined as follows.

Definition 2 (Set of explicit inversive points of period T). Let $0 \neq \theta \in \mathbb{F}_q$ be an element of multiplicative order T (hence $T|(q-1)$) and $S \subseteq \mathbb{F}_q$ be of cardinality $1 \leq s \leq q$. Then we define

$$\mathcal{S} = \{\mathbf{z}_0, \dots, \mathbf{z}_{T-1}\} = \{(\overline{\theta^n + v})_{v \in S} : n = 0, \dots, T-1\} \subset \mathbb{F}_q^s \quad (4)$$

of size T and derive $\mathcal{P}_s = \{\mathbf{x}_0, \dots, \mathbf{x}_{T-1}\} \in [0, 1)^s$ from \mathcal{S} by (1) and (2). We remark that in this case $N = |\mathcal{P}_s| = T$ and T divides $q-1$.

We introduce some notation.

- For $s \in \mathbb{N}$ put $[s] := \{1, 2, \dots, s\}$.
- For a vector $\mathbf{x} = (x_1, x_2, \dots, x_s)$ and for a nonempty $\mathbf{u} \subseteq [s]$, let $\mathbf{x}_{\mathbf{u}}$ be the projection of \mathbf{x} onto the components whose index belongs to \mathbf{u} , that is, for $\mathbf{u} = \{u_1, u_2, \dots, u_w\}$ with $u_1 < u_2 < \dots < u_w$ we have $\mathbf{x}_{\mathbf{u}} = (x_{u_1}, x_{u_2}, \dots, x_{u_w})$.
- Let ψ denote the canonical additive character of \mathbb{F}_q .
- For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^s$ let $\mathbf{x} \cdot \mathbf{y} \in \mathbb{F}_q$ denote their standard inner product.

Now, we are ready to state the first character sum bound.

Lemma 1. *Let $\mathcal{S} = \{z_0, \dots, z_{q-1}\}$ be given by (3) and let $\emptyset \neq \mathbf{u} \subseteq [s]$. Then we have*

$$\max_{\mathbf{w} \in \mathbb{F}_q^{|\mathbf{u}|} \setminus \{\mathbf{0}\}} \left| \sum_{n=0}^{q-1} \psi(\mathbf{w} \cdot \mathbf{z}_{n,\mathbf{u}}) \right| \leq (2|\mathbf{u}| - 2)q^{1/2} + |\mathbf{u}| + 1.$$

Proof. Note that the sums to be estimated are of the form

$$S_q := \sum_{n=0}^{q-1} \psi(\mathbf{w} \cdot \mathbf{z}_{n,\mathbf{u}}) = \sum_{u \in \mathbb{F}_q} \psi \left(\sum_{i=1}^{|\mathbf{u}|} w_i \overline{u + v_i} \right)$$

for some $(w_1, \dots, w_{|\mathbf{u}|}) \in \mathbb{F}_q^{|\mathbf{u}|} \setminus \{\mathbf{0}\}$ and $(v_1, \dots, v_{|\mathbf{u}|}) \in \mathbb{F}_q^{|\mathbf{u}|}$ with pairwise distinct coordinates $v_i \neq v_j$ if $i \neq j$.

We proceed as in the proof of [19, Theorem 1]. We have

$$|S_q| \leq |\mathbf{u}| + \left| \sum_{u \in \mathbb{F}_q, g(u) \neq 0} \psi \left(\frac{f(u)}{g(u)} \right) \right|,$$

where

$$f(x) = \sum_{i=1}^{|\mathbf{u}|} w_i \prod_{\substack{j=1 \\ j \neq i}}^{|\mathbf{u}|} (x + v_j)$$

and

$$g(x) = \prod_{j=1}^{|\mathbf{u}|} (x + v_j).$$

Since at least one of the w_i is non-zero and the v_i are distinct, we have $f(-v_i) = w_i \prod_{j \neq i} (v_j - v_i) \neq 0$ and f is not the zero polynomial. Since $\deg(f) < \deg(g)$, by [19, Lemma 2] the rational function $\frac{f(x)}{g(x)}$ is not of the form $A^p - A$ with some rational function over the algebraic closure of \mathbb{F}_q . Hence, we can apply a bound of Moreno and Moreno [17, Theorem 2] (see also [19, Lemma 1]) and the result follows. \square

For the second point set, which was essentially studied in [3, 26], we also give an analogous character sum bound.

Lemma 2. *Let $\{z_0, \dots, z_{T-1}\}$ be given by (4) of size T and let $\emptyset \neq \mathbf{u} \subseteq [s]$. Then we have*

$$\max_{\mathbf{w} \in \mathbb{F}_q^{|\mathbf{u}|} \setminus \{\mathbf{0}\}} \left| \sum_{n=0}^{T-1} \psi(\mathbf{w} \cdot \mathbf{z}_{n,\mathbf{u}}) \right| \leq 2|\mathbf{u}|q^{1/2} + |\mathbf{u}|.$$

Proof. The proof is analogous to the proof of [26, Theorem 1]. \square

3 The weighted star discrepancy

For an N -element point set \mathcal{P}_s in $[0, 1]^s$ the *local discrepancy* $\Delta_{\mathcal{P}_s}$ is defined as

$$\Delta_{\mathcal{P}_s}(\boldsymbol{\alpha}) = \frac{|\mathcal{P}_s \cap [\mathbf{0}, \boldsymbol{\alpha})|}{N} - \text{Volume}([\mathbf{0}, \boldsymbol{\alpha}))$$

for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s) \in [0, 1]^s$. The *star discrepancy* is then the L_∞ -norm of the local discrepancy,

$$D_N^*(\mathcal{P}_s) = \|\Delta_{\mathcal{P}_s}\|_{L_\infty}.$$

We consider the weighted star discrepancy. The study of weighted discrepancy has been initiated by Sloan and Woźniakowski [24] in 1998 in order to overcome the curse of dimensionality. Their basic idea was to introduce a set of weights $\boldsymbol{\gamma} = \{\gamma_{\mathbf{u}} : \emptyset \neq \mathbf{u} \subseteq [s]\}$ which consists of non-negative real numbers $\gamma_{\mathbf{u}}$. A simple choice of weights are so-called product weights $(\gamma_j)_{j \geq 1}$, where $\gamma_{\mathbf{u}} = \prod_{j \in \mathbf{u}} \gamma_j$. In this case, the weight γ_j is associated with the variable x_j .

Definition 3 (Weighted star discrepancy). For given weights $\boldsymbol{\gamma}$ and for a point set \mathcal{P}_s in $[0, 1]^s$ the *weighted star discrepancy* is defined as

$$D_{N, \boldsymbol{\gamma}}^*(\mathcal{P}_s) = \max_{\emptyset \neq \mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}} \sup_{\boldsymbol{\alpha} \in [0, 1]^s} |\Delta_{\mathcal{P}_s}((\boldsymbol{\alpha}_{\mathbf{u}}, \mathbf{1}))|,$$

where for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s) \in [0, 1]^s$ and for $\mathbf{u} \subseteq [s]$ we put $(\boldsymbol{\alpha}_{\mathbf{u}}, \mathbf{1}) = (y_1, \dots, y_s)$ with

$$y_j = \begin{cases} \alpha_j & \text{if } j \in \mathbf{u}, \\ 1 & \text{if } j \notin \mathbf{u}. \end{cases}$$

Remark 1. Let \mathcal{F}_1 be the space of functions with finite norm

$$\|f\|_{\mathcal{F}_1} := \sum_{\mathbf{u} \subseteq [s]} \frac{1}{\gamma_{\mathbf{u}}} \int_{[0, 1]^{|\mathbf{u}|}} \left| \frac{\partial^{|\mathbf{u}|} f}{\partial \mathbf{x}_{\mathbf{u}}}(\mathbf{x}_{\mathbf{u}}, \mathbf{1}) \right| d\mathbf{x}_{\mathbf{u}},$$

where for $\mathbf{u} = \emptyset$ we put $\frac{\partial^{|\mathbf{u}|} f}{\partial \mathbf{x}_{\mathbf{u}}}(\mathbf{x}_{\mathbf{u}}, \mathbf{1}) = f(1, 1, \dots, 1)$. Then it was shown in [24, p. 12] that the weighted star discrepancy of a point set \mathcal{P}_s is an upper bound for the worst-case error of the QMC rule Q_N based on \mathcal{P}_s , that is,

$$e(Q_N, \mathcal{F}_1) \leq D_{N, \boldsymbol{\gamma}}^*(\mathcal{P}_s).$$

It is well known that there is a close connection between discrepancy and character sums. In discrepancy theory such relations are known under the name “*Erdős-Turán-Koksma inequalities*”. One particular instance of an Erdős-Turán-Koksma inequality is given in the following lemma which is perfectly suited for our applications. Before stating the result, we introduce the following auxiliary function. For $q = p^k$, we define

$$\mathcal{T}(q, s) = \begin{cases} \left(\frac{k}{2} + 1\right)^s & \text{if } p = 2, \\ \left(\frac{2}{\pi} \log p + \frac{7}{5}\right)^s k^s & \text{if } p > 2. \end{cases} \quad (5)$$

The result is the following:

Lemma 3. For $q = p^k$ and $\mathbf{z}_0, \dots, \mathbf{z}_{N-1} \in \mathbb{F}_q^s$, let $\mathcal{P}_s = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ be the N -element point set defined by (1) and (2). Then we have

$$D_N^*(\mathcal{P}_s) \leq \frac{s}{q} + \frac{\mathcal{T}(q, s)}{N} \max_{\mathbf{w} \in \mathbb{F}_q^s \setminus \{\mathbf{0}\}} \left| \sum_{n=0}^{N-1} \psi(\mathbf{w} \cdot \mathbf{z}_n) \right|,$$

where $\mathcal{T}(q, s)$ is defined as in (5).

Proof. For a non-zero $s \times k$ matrix $H = (h_{ij})$ with entries $h_{ij} \in (-p/2, p/2) \cap \mathbb{Z}$ we define the exponential sum

$$S_N(H) = \sum_{n=0}^{N-1} \exp \left(\frac{2\pi \mathbf{i}}{p} \sum_{i=1}^s \sum_{j=1}^k h_{ij} c_{n,i}^{(j)} \right),$$

where the $c_{n,i}^{(j)} \in \mathbb{F}_p$ are defined by (1) and where $\mathbf{i} = \sqrt{-1}$. By a general discrepancy bound taken from [18, Theorem 3.12 and Lemma 3.13] we get

$$D_N^*(\mathcal{P}_s) \leq \frac{s}{q} + \frac{\mathcal{T}(q, s)}{N} \max_{H \neq 0} |S_N(H)|,$$

where the maximum is extended over all non-zero matrices H with entries $h_{ij} \in (-p/2, p/2) \cap \mathbb{Z}$.

Let $\{\delta_1, \dots, \delta_k\}$ be the dual basis of the given ordered basis $\{\beta_1, \dots, \beta_k\}$ of \mathbb{F}_q over \mathbb{F}_p , that is,

$$\text{Tr}(\delta_j \beta_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where Tr denotes the trace function from \mathbb{F}_q to \mathbb{F}_p . For any basis, there exists a dual basis and this basis is unique, see [16, p. 55] for a proof. Then we have

$$c_{n,i}^{(j)} = \text{Tr}(\delta_j z_{n,i}) \quad \text{for } 1 \leq j \leq k, \ 1 \leq i \leq s, \text{ and } n \in \mathbb{N}_0,$$

where $\mathbf{z}_n = (z_{n,1}, \dots, z_{n,s})$. Therefore

$$\begin{aligned} S_N(H) &= \sum_{n=0}^{N-1} \exp \left(\frac{2\pi \mathbf{i}}{p} \sum_{i=1}^s \sum_{j=1}^k h_{ij} \text{Tr}(\delta_j z_{n,i}) \right) \\ &= \sum_{n=0}^{N-1} \exp \left(\frac{2\pi \mathbf{i}}{p} \text{Tr} \left(\sum_{i=1}^s \sum_{j=1}^k h_{ij} \delta_j z_{n,i} \right) \right) \\ &= \sum_{n=0}^{N-1} \psi \left(\sum_{i=1}^s \sum_{j=1}^k h_{ij} \delta_j z_{n,i} \right). \end{aligned}$$

Put

$$\mathbf{w} = (w_1, \dots, w_s) \in \mathbb{F}_q^s \text{ with } w_i = \sum_{j=1}^k h_{ij} \delta_j \text{ for } i = 1, \dots, s.$$

Since H is not the zero matrix and $\{\delta_1, \dots, \delta_k\}$ is a basis of \mathbb{F}_q over \mathbb{F}_p , it follows that \mathbf{w} is not the zero vector. This fact finishes the proof. \square

Now we extend the star discrepancy estimate from Lemma 3 to the weighted star discrepancy:

Lemma 4. *For $\mathbf{z}_0, \dots, \mathbf{z}_{N-1} \in \mathbb{F}_q^s$ let $\mathcal{P}_s = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ be the N -element point set defined by (1) and (2). Then we have*

$$D_{N,\gamma}^*(\mathcal{P}_s) \leq \max_{\emptyset \neq \mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}} \left(\frac{|\mathbf{u}|}{q} + \frac{\mathcal{T}(q, |\mathbf{u}|)}{N} \max_{\mathbf{w} \in \mathbb{F}_q^{|\mathbf{u}|} \setminus \{\mathbf{0}\}} \left| \sum_{n=0}^{N-1} \psi(\mathbf{w} \cdot \mathbf{z}_{n,\mathbf{u}}) \right| \right).$$

where $\mathbf{z}_{n,\mathbf{u}} \in \mathbb{F}_q^{|\mathbf{u}|}$ is the projection of \mathbf{z}_n to the coordinates indexed by \mathbf{u} .

Proof. The result follows immediately from Lemma 3 together with the fact that

$$D_{N,\gamma}^*(\mathcal{P}_s) \leq \max_{\emptyset \neq \mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}} D_N^*(\mathcal{P}_{\mathbf{u}}),$$

where $\mathcal{P}_{\mathbf{u}}$ consists of the points from \mathcal{P}_s projected onto the components whose indices belong to \mathbf{u} . \square

The previous lemma gives us our first main result.

Theorem 1. *For the point set \mathcal{P}_s defined as in Definition 1 the following bound holds:*

$$D_{q,\gamma}^*(\mathcal{P}_s) \leq \max_{\emptyset \neq \mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}} |\mathbf{u}| \left(\frac{1}{q} + \frac{3\mathcal{T}(q, |\mathbf{u}|)}{q^{1/2}} \right).$$

For the point set \mathcal{P}_s defined as in Definition 2 the following bound holds:

$$D_{T,\gamma}^*(\mathcal{P}_s) \leq \max_{\emptyset \neq \mathbf{u} \subseteq [s]} \gamma_{\mathbf{u}} |\mathbf{u}| \left(\frac{1}{q} + \frac{3\mathcal{T}(q, |\mathbf{u}|)q^{1/2}}{T} \right).$$

Proof. The result follows from Lemma 4 and Lemmas 1 and 2. \square

We point out that the discrepancy estimate from Theorem 1 holds for every choice of weights. Also, it is important to remark that point sets defined by (4) are more flexible in terms of size. It is easy to check that for any prime p and T not divisible by p , there exists $q = p^k$, such that T divides $q - 1$. This k is the multiplicative order of p modulo T . Since the multiplicative group of \mathbb{F}_q is cyclic there is an element $\theta \in \mathbb{F}_q$ of order T .

Tractability. For a recent overview of results concerning tractability properties of the weighted star discrepancy we refer to [7, 8].

Important Remark 2. Unfortunately the proof of [8, Theorem 3.2 (ii)] (also [7, Theorem 7(2)]) is not correct and hence this part of the theorem must be discarded. All other parts of these papers are correct.

We now restrict ourselves to product weights and present a condition on the weights for strong polynomial tractability.

Theorem 2. Let \mathcal{P}_s be the point set from Definition 1. Assume that for an ordered sequence of weights $\gamma = (\gamma_j)_{j \geq 1}$ with $\gamma_1 \geq \gamma_2 \geq \dots$, there is a $0 \leq \delta < 1/2$ such that

$$\limsup_{j \rightarrow \infty} j\gamma_j < \frac{\delta}{3}. \quad (6)$$

Then there is a constant $c_{\gamma, \delta} > 0$, which depends only on γ and δ but not on s such that for all $1 \leq s \leq q$ we have

$$D_{q, \gamma}^*(\mathcal{P}_s) \leq \frac{c_{\gamma, \delta}}{q^{1/2-\delta}}.$$

If $\limsup_{j \rightarrow \infty} j\gamma_j = 0$, then the result holds for all $\delta > 0$.

Proof. We show the result for $p > 2$ and when the weights satisfy (6) only. The other cases are proven in a similar way. From Theorem 1, the fact that $r \leq 2^r$ for $r \geq 1$ and using the ordering of γ we have

$$\begin{aligned} D_{q, \gamma}^*(\mathcal{P}_s) &\leq C^{(1)} \max_{r=1, \dots, s} \frac{r \prod_{j=1}^r (\gamma_j k (\frac{2}{\pi} \log p + \frac{7}{5}))}{q^{1/2}} \\ &\leq C^{(1)} \max_{r=1, \dots, s} \frac{\prod_{j=1}^r (2\gamma_j k (\frac{2}{\pi} \log p + \frac{7}{5}))}{q^{1/2}}. \end{aligned}$$

Notice that $2k(\frac{2}{\pi} \log p + \frac{7}{5})$ can also be bounded by $c \log q$ for some $0 < c < 3$. Now, let ℓ be the largest integer such that $c\gamma_\ell \log q > 1$. Then we have

$$D_{q, \gamma}^*(\mathcal{P}_s) \leq C^{(1)} \frac{\prod_{j=1}^\ell (c\gamma_j \log q)}{q^{1/2}}.$$

The condition $\limsup_{j \rightarrow \infty} j\gamma_j < \delta/3$ implies that there is an $L > 0$ such that $j\gamma_j < \delta/3$ for all $j \geq L$. Without loss of generality we may assume that $\ell \geq L$. (Otherwise, if $\ell < L$, consider a new weight sequence $\gamma' = (\gamma'_j)_{j \geq 1}$ with $\gamma'_j = \gamma_j$ for all $j \in \{1, \dots, \ell\} \cup \{L, L+1, \dots\}$ and $\gamma'_j = \gamma_\ell$ for $j \in \{\ell+1, \dots, L-1\}$, and hence $\gamma_j \leq \gamma'_j$ for all $j \geq 1$).

For $r \in \mathbb{N}$ let

$$c_r = \prod_{j=1}^r (c\gamma_j \log q),$$

so we have

$$\frac{c_\ell}{c_{\ell-1}} = c\gamma_\ell \log q < \frac{c\delta}{3\ell} \log q.$$

By the definition of ℓ we have $c_{\ell-1} < c_\ell$, hence

$$1 < \frac{c\delta}{3\ell} \log q,$$

which implies $\ell < c\delta(\log q)/3$, or $\ell \leq \lfloor c\delta(\log q)/3 \rfloor$.

Therefore, there is a constant $C_\gamma^{(2)} > 0$ such that

$$\begin{aligned} \prod_{j=1}^\ell (c\gamma_j \log q) &= \prod_{j=1}^{L-1} (c\gamma_j \log q) \prod_{j=L}^\ell (c\gamma_j \log q) \\ &\leq C_\gamma^{(2)} (c \log q)^{L-1} \prod_{j=L}^{\lfloor c\delta(\log q)/3 \rfloor} \frac{c\delta \log q}{3j}. \end{aligned}$$

Let $x := c\delta(\log q)/3$. Then

$$\begin{aligned}
\prod_{j=1}^{\ell} (c\gamma_j \log q) &\leq C_{\gamma}^{(2)} \left(\frac{c \log q}{x} \right)^{L-1} ((L-1)!) \frac{x^{\lfloor x \rfloor}}{\lfloor x \rfloor!} \\
&\leq C_{\gamma}^{(2)} \left(\frac{3}{\delta} \right)^{L-1} ((L-1)!) e^x \\
&= C_{\gamma, \delta}^{(3)} q^{c\delta/3} \\
&< C_{\gamma, \delta}^{(3)} q^{\delta},
\end{aligned}$$

where $C_{\gamma, \delta}^{(3)} = C_{\gamma}^{(2)} \left(\frac{3}{\delta} \right)^{L-1} ((L-1)!) (note that L only depends on γ).$ This implies

$$D_{q, \gamma}^*(\mathcal{P}_s) \leq C^{(1)} \frac{3C_{\gamma, \delta}^{(3)}}{q^{1/2-\delta}}$$

and finishes the proof. \square

Remark 3. Note that $\sum_{j=1}^{\infty} \gamma_j < \infty$, together with the monotonicity $\gamma_1 \geq \gamma_2 \geq \dots$ implies $\limsup_{j \rightarrow \infty} j\gamma_j = 0$. To see this let $\varepsilon > 0$. From $\sum_{j=1}^{\infty} \gamma_j < \infty$ it follows with the Cauchy condensation test that also $\sum_{k=0}^{\infty} 2^k \gamma_{2^k} < \infty$. In particular, $2^k \gamma_{2^k} \rightarrow 0$ for $k \rightarrow \infty$. This means that $\gamma_{2^k} \leq \varepsilon/2^{k+1}$ for k large enough. Thus, for large enough j with $2^k \leq j < 2^{k+1}$ we obtain

$$\gamma_j \leq \gamma_{2^k} \leq \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{j}.$$

In particular, for j large enough we have $j\gamma_j < \varepsilon$. This implies that

$$\limsup_{j \rightarrow \infty} j\gamma_j = 0.$$

Of course the converse is not true in general (for example, $\gamma_j = 1/(j \log j)$).

Corollary 1. *With the notation and conditions as in Theorem 2, in particular $\limsup_{j \rightarrow \infty} \gamma_j < \delta/3$, the weighted star discrepancy (or, equivalently, integration in \mathcal{F}_1) is strongly polynomially tractable with ε -exponent at most $2/(1-2\delta)$.*

Proof. For $\varepsilon > 0$ let $M := \lceil (c_{\gamma, \delta} \varepsilon^{-1})^{2/(1-2\delta)} \rceil$. Let q be the smallest prime power which is greater or equal to M . According to the Postulate of Bertrand we have $q < 2M$. Then we have $D_{q, \gamma}^*(\mathcal{P}_s) \leq \varepsilon$ and hence the information complexity satisfies

$$N(\varepsilon, s) \leq q \leq 2M = 2 \lceil (c_{\gamma, \delta} \varepsilon^{-1})^{2/(1-2\delta)} \rceil.$$

This means that we have strong polynomial tractability. \square

For the proof of Corollary 1 it is enough to use the construction of Definition 1 with a prime q . However, from a practical point of view the construction of Definition 1 with any prime power q and the construction of Definition 2 provide more flexibility. In particular the case $q = 2^r$ can be efficiently implemented using (optimal) normal bases and the Itoh-Tsujii inversion algorithm, see [14, Chapter 3] and [13], respectively.

4 Integration of Hölder continuous, absolutely convergent Fourier series and cosine series

Absolutely convergent Fourier series. For $f \in L_2([0, 1]^s)$ and $\mathbf{h} \in \mathbb{Z}^s$ we define the \mathbf{h} th Fourier coefficient of f as $\widehat{f}(\mathbf{h}) = \int_{[0,1]^s} f(\mathbf{x}) e^{-2\pi i \mathbf{h} \cdot \mathbf{x}} d\mathbf{x}$. Then we can associate to f its Fourier series

$$f(\mathbf{x}) \sim \sum_{\mathbf{h} \in \mathbb{Z}^s} \widehat{f}(\mathbf{h}) e^{2\pi i \mathbf{h} \cdot \mathbf{x}}. \quad (7)$$

Let $\alpha \in (0, 1]$ and $t \in [1, \infty]$. Similarly to [4] we consider the norm

$$\|f\|_{K_{\alpha,t}} = \sum_{\mathbf{u} \subseteq [s]} |\mathbf{u}| \sum_{\mathbf{k}_{\mathbf{u}} \in \mathbb{Z}_{*}^{|\mathbf{u}|}} |\widehat{f}(\mathbf{k}_{\mathbf{u}}, \mathbf{0})| + \|f\|_{H_{\alpha,t}},$$

where $\mathbb{Z}_{*} = \mathbb{Z} \setminus \{0\}$ and where

$$\|f\|_{H_{\alpha,t}} = \sup_{\mathbf{x}, \mathbf{x}+\mathbf{h} \in [0,1]^s} \frac{|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|}{\|\mathbf{h}\|_{\ell_t}^{\alpha}},$$

is the Hölder semi-norm where $\|\cdot\|_{\ell_t}$ denotes the norm in ℓ_t .

We define the following sub-class of the Wiener algebra

$$K_{\alpha,t} := \{f \in L_2([0, 1]^s) : f \text{ is one-periodic and } \|f\|_{K_{\alpha,t}} < \infty\}.$$

The choice of t will influence the dependence on the dimension of the worst-case error upper bound. As in [4] we remark that for any $f \in K_{\alpha,t}$ the Fourier series (7) of f converges to f at every point $\mathbf{x} \in [0, 1]^s$. This follows directly from [25, Corollary 1.8, p. 249], using that f is continuous since it satisfies a Hölder condition, i.e. $\|f\|_{H_{\alpha,t}} < \infty$. More information on $K_{\alpha,t}$ can be found in [4].

Theorem 3. Let \mathcal{P}_s be the point set from Definition 1 with $k = 1$ and $q = p = N$ and let Q_N be the QMC rule based on \mathcal{P}_s . Then for $\alpha \in (0, 1]$ and $t \in [1, \infty]$ we have

$$e(Q_N, K_{\alpha,t}) \leq \max\left(\frac{3}{\sqrt{N}}, \frac{s^{\alpha/t}}{N^{\alpha}}\right).$$

In particular, if $t = \infty$ we have

$$e(Q_N, K_{\alpha,\infty}) \leq \frac{3}{N^{\min(\alpha, 1/2)}}.$$

Proof. For $f \in K_{\alpha,t}$ we have

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right| = \left| \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} \widehat{f}(\mathbf{k}) \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i \mathbf{k} \cdot \mathbf{x}_n} \right| \\ & \leq \sum_{\substack{\mathbf{k} \in \mathbb{Z}^s \\ N \nmid \mathbf{k}}} |\widehat{f}(\mathbf{k})| \frac{1}{N} \left| \sum_{n=0}^{N-1} e^{\frac{2\pi i}{p} \mathbf{k} \cdot \mathbf{c}_n} \right| + \sum_{\substack{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\} \\ N \mid \mathbf{k}}} |\widehat{f}(\mathbf{k})| \\ & = \sum_{\emptyset \neq \mathbf{u} \subseteq [s]} \sum_{\substack{\mathbf{k}_{\mathbf{u}} \in \mathbb{Z}_{*}^{|\mathbf{u}|} \\ N \nmid \mathbf{k}_{\mathbf{u}}}} |\widehat{f}((\mathbf{k}_{\mathbf{u}}, \mathbf{0}))| \frac{1}{N} \left| \sum_{n=0}^{N-1} e^{\frac{2\pi i}{p} \mathbf{k}_{\mathbf{u}} \cdot \mathbf{c}_{n,\mathbf{u}}} \right| + \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} |\widehat{f}(N\mathbf{k})|, \end{aligned}$$

where $N \mid \mathbf{k}$ if all coordinates of \mathbf{k} are divisible by N and $N \nmid \mathbf{k}$ otherwise. Now we apply Lemma 1 to the first sum and [4, Lemma 1] to the second sum and obtain

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x} \right| \\ & \leq \frac{3}{\sqrt{N}} \sum_{\emptyset \neq \mathbf{u} \subseteq [s]} |\mathbf{u}| \sum_{\substack{\mathbf{k}_{\mathbf{u}} \in \mathbb{Z}_*^{|\mathbf{u}|} \\ N \nmid \mathbf{k}_{\mathbf{u}}}} |\widehat{f}((\mathbf{k}_{\mathbf{u}}, \mathbf{0}))| + \frac{s^{\alpha/t}}{N^\alpha} |f|_{H_{\alpha,t}} \\ & \leq \max \left(\frac{3}{\sqrt{N}}, \frac{s^{\alpha/t}}{N^\alpha} \right) \|f\|_{K_{\alpha,t}}. \end{aligned}$$

The result follows. \square

Corollary 2. *Integration in $K_{\alpha,\infty}$ is strongly polynomially tractable with ε -exponent at most $\max(\frac{1}{\alpha}, 2)$.*

Proof. The proof is similar to the one of Corollary 1. \square

Absolutely convergent cosine series. So far we required that the functions are periodic. Now we show how we can get rid of this assumption. Let us consider cosine series instead of classical Fourier series.

The cosine system $\{\cos(k\pi x) : k \in \mathbb{N}_0\}$ forms a complete orthogonal basis of $L_2([0, 1])$. To get an ONB we need to normalize it. Hence we define

$$\sigma_k(x) := \begin{cases} 1 & \text{if } k = 0, \\ \sqrt{2} \cos(k\pi x) & \text{if } k \in \mathbb{N}, \end{cases}$$

then the system

$$\{\sigma_k : k \in \mathbb{N}_0\}$$

forms an ONB of $L_2([0, 1])$. For $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ and $\mathbf{x} = (x_1, \dots, x_s) \in [0, 1]^s$ define

$$\sigma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^s \sigma_{k_j}(x_j).$$

The $\sigma_{\mathbf{k}}$ with $\mathbf{k} \in \mathbb{N}_0^s$ constitute an ONB of $L_2([0, 1]^s)$.

To an L_2 -function g we can associate the cosine series

$$g(\mathbf{x}) \sim \sum_{\mathbf{k} \in \mathbb{N}_0^s} \widetilde{g}(\mathbf{k}) \sigma_{\mathbf{k}}(\mathbf{x})$$

with cosine coefficients $\widetilde{g}(\mathbf{k}) = \int_{[0,1]^s} g(\mathbf{x}) \sigma_{\mathbf{k}}(\mathbf{x}) \, d\mathbf{x}$.

In order to apply the results for Fourier series to cosine series we need the tent transformation $\phi : [0, 1] \rightarrow [0, 1]$ given by

$$\phi(x) = 1 - |2x - 1|.$$

For vectors \mathbf{x} the tent transformed point $\phi(\mathbf{x})$ is understood component wise. The tent transformation is a Lebesgue measure preserving map and we have

$$\int_{[0,1]^s} g(\mathbf{x}) \, d\mathbf{x} = \int_{[0,1]^s} g(\phi(\mathbf{x})) \, d\mathbf{x}.$$

Define the norm

$$\|g\|_{C_{\alpha,t}} = \sum_{u \subseteq [s]} |u| 2^{|u|/2} \sum_{\mathbf{k}_u \in \mathbb{N}^{|u|}} |\tilde{g}((\mathbf{k}_u, \mathbf{0}))| + 2^\alpha |g|_{H_{\alpha,t}}$$

and let

$$C_{\alpha,t} = \{g \in L_2([0, 1]^s) : \|g\|_{C_{\alpha,t}} < \infty\}.$$

For $g \in L_2([0, 1]^s)$ we have that the function $f = g \circ \phi$ is one-periodic and

$$\|f\|_{K_{\alpha,t}} = \|g\|_{C_{\alpha,t}}.$$

The cosine series of g converges point-wise and absolute to g for all points in $[0, 1]^s$.

Now we consider integration of functions from $C_{\alpha,t}$: For a point set $\mathcal{P}_s = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ let $\mathcal{Q} = \{\phi(\mathbf{x}_0), \dots, \phi(\mathbf{x}_{N-1})\}$ be the tent transformed version of \mathcal{P}_s . Let Q_N be a QMC rule based on \mathcal{P}_s . Then we denote by Q_N^ϕ the QMC rule based on \mathcal{Q} .

As in [4, Proof of Theorem 2] we have the identity of worst-case errors in $K_{\alpha,t}$ and $C_{\alpha,t}$ when we switch from a QMC rule to the tent transformed version of this rule, namely

$$e(Q_N, K_{\alpha,t}) = e(Q_N^\phi, C_{\alpha,t}).$$

With this identity we can transfer the results for periodic functions to not necessarily periodic ones.

Corollary 3. *Let \mathcal{P}_s be the point set from Definition 1 with $k = 1$ and $q = p = N$ and let Q_N^ϕ be the tent transformed version of the QMC rule based on \mathcal{P}_s . Then for $\alpha \in (0, 1]$ and $t \in [1, \infty]$ we have*

$$e(Q_N^\phi, C_{\alpha,t}) \leq \max\left(\frac{3}{\sqrt{N}}, \frac{s^{\alpha/t}}{N^\alpha}\right).$$

In particular, if $t = \infty$ we have

$$e(Q_N^\phi, C_{\alpha,\infty}) \leq \frac{3}{N^{\min(\alpha, 1/2)}}.$$

Corollary 4. *Integration in $C_{\alpha,\infty}$ is strongly polynomially tractable with ε -exponent at most $\max(\frac{1}{\alpha}, 2)$.*

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